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Sharp Bounds for Ground State Eigenfunctions on Domains with Horns and Cusps

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We prove a lower bound on the ground state eigenfunction of the Dirichlet Laplacian for horn-shaped and cusped domains. The novelty of the method is that for horn-shaped domains it implies a new result for dimension two, and more importantly, is valid for higher dimensions. Moreover, the same technique applies just as easily to cusps. Finally, we construct an example to show that the estimates are sharp.

1. INTRODUCTION

There has long been interest in properties of the ground state eigenfunction ϕ of the Dirichlet Laplacian on various domains $\Omega \subset \mathbf{R}^N$. As shown by Davies and Simon [10], there is a certain class of domains for which the heat semigroup is "intrinsically ultracontractive." For a domain to enjoy this property, it suffices that the ground state eigenfunction satisfy certain exponential decay estimates. In return, one obtains that all the other eigenfunctions and the heat kernel itself have bounds in terms of ϕ.

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In addition, control of ϕ also has implications for the Green's function. Quite often, precise estimates of ϕ can be made in terms of the geometry of the domain and so the results are easy to apply in practice. Partly because of this, the notion of intrinsic ultracontractivity has come to play an important role in the study of Schrödinger operators and second order elliptic operators. Intrinsic ultracontractivity is also closely related to the parabolic boundary Harnack principle and to the lifetime of conditioned brownian motion. For more details we refer the reader to [1, 6].

The purpose of this paper is twofold. First of all, in Theorem 1 we establish sharp lower bounds on ϕ for a class of rotationally symmetric horn-shaped domains. For $N \geq 3$ this result is new, and for $N = 2$, the estimate improves a result of Bañuelos and Davis in certain cases [4, 5]. In fact, the initial motivation for this work originated in the search for a method to prove estimates for $N \geq 3$; although the methods of [4, 5], and more recently [2], produce sharp results, they rely on methods from conformal analysis. Out of our approach, one sees that the bound in Theorem 1 can easily be transferred to a simple outward cusp. This is the content of Theorem 2. The novelty of this result is that one now has estimates on a domain which contains a boundary point that is not non-tangentially accessible. Later in this introduction we outline how Theorem 2 and the maximum principle can be used to give estimates for arbitrary domains with outward cusps. Theorem 3 and its Corollary justify our claims that Theorems 1 and 2 are sharp. The results in this paper are those announced in [12, 13].

We now discuss the results in greater detail. Throughout the paper our horn-shaped domains are defined as

$$D_\rho = \{(x, y) \in \mathbf{R} \times \mathbf{R}^{N-1}\colon x > 0, |y| < \rho(x)\}, \tag{1.1}$$

where it is assumed that $\rho\colon [0,\infty) \to (0,\infty)$ satisfies

(A1) $\rho(x) \downarrow 0$ as $x \to \infty$

(A2) ρ is differentiable on $[0,\infty)$ and $|\rho'(x)| \downarrow 0$ as $x \to \infty$.

Assumption (A2) is essential as the proof is based on a sequence of overlapping cones in D_ρ; if ρ wiggles its way to zero these cones do not remain inside the horn.

We let $-\Delta$ denote the Dirichlet Laplacian for whatever domain is currently under discussion and γ is the ground state eigenvalue of the unit ball in $\mathbf{R}^{N-1}$.

THEOREM 1. *For a horn-shaped region D_ρ, the following lower bound on the ground state eigenfunction holds*:

(i) *If $N \geq 3$, then there exists $c \geq 1$ such that for all $x \geq 1$,*

$$\phi(x,0) \geq c^{-1} \exp\left\{ -\sqrt{\gamma} \int_1^x \frac{dt}{\rho(t)} + \left(\frac{N}{2} - 1 \right) \int_1^x \frac{|\rho'(t)|}{\rho(t)}\, dt \right.$$
$$\left. + O\left(\int_1^x \frac{\rho'(t)^2}{\rho(t)}\, dt \right) \right\}.$$

(ii) *If $N = 2$, then there exists $c \geq 1$ such that for all $x \geq 1$*

$$\phi(x,0) \geq c^{-1} \exp\left\{ -\frac{\pi}{2} \int_1^x \frac{dt}{\rho(t)} - \frac{\pi}{6} \int_1^x \frac{\rho'(t)^2}{\rho(t)}\, dt \right.$$
$$\left. + \frac{\pi}{48} \int_1^x \frac{\rho'(t)^4}{\rho(t)}\, dt + O\left(\int_1^x \frac{\rho'(t)^8}{\rho(t)}\, dt \right) \right\}.$$

We recall that Bañuelos and Davis [4, 5] proved that if $\rho: [0,\infty) \to (0,\infty)$ is a differentiable decreasing function satisfying (A1) and $\|\rho'\|_\infty < \infty$, then for $N = 2$ there exists $c \geq 1$ such that for all $x \geq 1$

$$\phi(x,0) \geq c^{-1} \exp\left\{ -\frac{\pi}{2} \int_1^x \frac{dt}{\rho(t)} - \frac{\pi}{6} \int_1^x \frac{\rho'(t)^2}{\rho(t)}\, dt \right\}.$$

On the surface, it appears that Theorem 1 is sharper as soon as (A2) is satisfied. However, this is only the case for ρ that additionally satisfy

$$\int_1^\infty \frac{\rho'(t)^4}{\rho(t)}\, dt = \infty. \tag{1.2}$$

In Theorem 3 we will show that such functions do exist.

Before presenting Theorem 2, we introduce the necessary assumptions and notation. We define a cusped domain

$$\Omega_\nu = \left\{ (x,y) \in \mathbf{R} \times \mathbf{R}^{N-1} : 0 < x < 1, |y| < \nu(x) \right\}, \tag{1.3}$$

where it is assumed that $\nu: (0,1] \to (0,\infty)$ is a strictly increasing differentiable function satisfying

(B1) $\lim_{x \downarrow 0} \nu(x) = 0$

(B2) $|\nu'(x)| \downarrow 0$ as $x \downarrow 0$.

THEOREM 2. *For a cusped domain* Ω_ν *the following lower bound on the ground state eigenfunction holds*:

(i) *If* $N \geq 3$, *then there exists* $c \geq 1$ *such that for all* $x \in (0, 1/2)$,

$$\phi(x,0) \geq c^{-1} \exp\left\{ -\sqrt{\gamma} \int_x^{1/2} \frac{dt}{\nu(t)} + \left(\frac{N}{2} - 1\right) \int_x^{1/2} \frac{|\nu'(t)|}{\nu(t)}\, dt + O\left(\int_x^{1/2} \frac{\nu'(t)^2}{\nu(t)}\, dt \right) \right\}.$$

(ii) *If* $N = 2$, *then there exists* $c \geq 1$ *such that for all* $x \in (0, 1/2)$

$$\phi(x,0) \geq c^{-1} \exp\left\{ -\frac{\pi}{2} \int_x^{1/2} \frac{dt}{\nu(t)} - \frac{\pi}{6} \int_x^{1/2} \frac{\nu'(t)^2}{\nu(t)}\, dt + \frac{\pi}{48} \int_x^{1/2} \frac{\nu'(t)^4}{\nu(t)}\, dt + O\left(\int_x^{1/2} \frac{\nu'(t)^8}{\nu(t)}\, dt \right) \right\}.$$

We now discuss how the estimate of Theorem 2 can be applied to arbitrary domains with cusps. First, let Ω be a domain satisfying an internal cone condition and let ξ be a boundary point of Ω. Given a positive harmonic function h with zero boundary value in a neighborhood of ξ, we wish to obtain a lower bound for h near ξ. A standard way to accomplish this is to fix a cone $K \subseteq \Omega$ with vertex ξ and a positive harmonic function h_K defined on K that has zero boundary value in a neighborhood of ξ. The maximum principle then implies that

$$h(x) \geq c h_K(x), \qquad \forall x \in K. \tag{1.4}$$

Since the decay rate of h_K near ξ can be computed precisely by separation of variables, (1.4) gives an explicit lower bound for h near ξ. Now if we remove the assumption that Ω satisfies an interior cone condition and suppose that ξ is not non-tangentially accessible, we can still obtain a lower bound for h near ξ: simply replace the cone with a cusp of the type discussed in Theorem 2. The explicit lower bound is then given by Theorem 2(i) and (ii).

As we mentioned in the discussion after Theorem 1, it is not immediately clear that there exist functions satisfying (A1), (A2), and (1.2).

THEOREM 3. *For any* $1 \le r \le 4$, *there exists a bounded, positive, differentiable function* f *on* $[1, \infty)$ *that satisfies* (A1), (A2), *and*

$$\int_1^\infty \frac{|f'(t)|^r}{f(t)}\, dt = \infty.$$

Moreover, Theorem 3 shows that Theorem 1 is sharp. For example, by rearranging (ii) we have that for all $x \ge 1$,

$$0 \le \frac{\exp\left\{-(\pi/2)\int_1^x (dt/\rho(t)) - (\pi/6)\int_1^x \left(\rho'(t)^2/\rho(t)\right) dt\right\}}{\phi(x,0)}$$
$$\le c \exp\left\{-\frac{\pi}{48}\int_1^x \frac{\rho'(t)^4}{\rho(t)}\, dt\right\}.$$

In the case where (1.2) holds, we have a precise estimate for the decay rate as $x \to \infty$. The following result is the analogue for Theorem 3:

COROLLARY 1. *For any* $1 \le r \le 4$, *there exists a bounded, positive, differentiable function* ν *on* $(0,1)$ *satisfying* (B1), (B2), *and*

$$\int_0^{1/2} \frac{|\nu'(t)|^r}{\nu(t)}\, dt = \infty. \tag{1.5}$$

We mention some recent work related to the results we present here. Bañuelos and van den Berg [3] have obtained lower bounds for the ground state eigenfunctions of a class of horn-shaped regions in $\mathbf{R}^N$, $N \ge 2$. However, the regions studied here and in [3] do not contain each other. But for certain horn-shaped regions in the overlap of these two classes (e.g., $\rho(x) = x^{-\alpha}$, α sufficiently large), Theorem 1 gives a better lower bound than the corresponding result in [3]. However, the lower bound of Bañuelos and van den Berg can be applied to certain horns which are not rotationally symmetric with respect to the x-axis.

The paper of Cranston and Li [7] studies these problems from a more general viewpoint. In particular, they determine bounds for the first eigenfunction on rotationally asymmetric twisted horns corresponding to the second order elliptic operator $L = \Delta + \alpha(x,y)\partial_x + \beta(x,y)\nabla_y + h(x,y)$. However, when specialized to the cases considered here, their results are not as sharp. Moreover, in their discussion of the rotationally symmetric case, they assume $\rho \in C^4$ and that (1.2) does not hold (see [7, 3.2]). Nonetheless, the significance of their work is that results in the spirit of this paper hold in greater generality.

The remainder of this paper is as follows. In Section 2 we present some lemmas required for the proof of Theorem 1, which is proved in Section 3. The proofs of Theorem 3 and its Corollary comprise Section 4.

2. PRELIMINARY LEMMAS

In this section we let $\beta \in (0, \pi/2)$ and let $-\Delta_\beta \geq 0$ be the Laplace–Beltrami operator defined on the geodesic ball

$$\left\{(x_1, x_2, \ldots, x_N) \in \mathbf{R}^N : \left|(x_1, \ldots, x_N)\right| = 1, \quad \cos\beta < x_1\right\}$$

of the unit sphere S^{N-1}. We let E_β be the ground state eigenvalue of $-\Delta_\beta$ and $\phi_\beta > 0$ be its corresponding eigenfunction. Then ϕ_β is a function of

$$\theta = \cos^{-1}(x_1)$$

only and satisfies

$$\begin{cases} \phi_\beta'' + (N-2)(\cot\theta)\phi_\beta' + E_\beta \phi_\beta = 0, & 0 < \theta < \beta \\ \phi_\beta(\beta) = \phi_\beta'(0) = 0. \end{cases} \tag{2.1}$$

Without loss of generality we shall assume that $\phi_\beta(0) = 1$. Put

$$\psi_\beta(t) = \phi_\beta(\beta t), \qquad 0 \leq t \leq 1.$$

Then (2.1) is equivalent to

$$\begin{cases} -a_\beta(t)^{-1} \dfrac{d}{dt}\left(a_\beta(t) \dfrac{d\psi_\beta}{dt}\right) = \lambda_\beta \psi_\beta(t), & 0 < t < 1 \\ \psi_\beta'(0) = \psi_\beta(1) = 0, \end{cases} \tag{2.2}$$

where $\lambda_\beta = \beta^2 E_\beta$ and

$$a_\beta(t) = \beta^{2-N} \sin^{N-2}(\beta t), \qquad 0 \leq t \leq 1.$$

We note that there exists $c_1 \geq 1$ such that

$$c_1^{-1} t^{N-2} \leq a_\beta(t) \leq c_1 t^{N-2}, \qquad 0 \leq t \leq 1 \tag{2.3}$$

for all $\beta \in (0, \pi/2)$. For each $\beta \in (0, \pi/2)$ we let $H_\beta \geq 0$ be the self-adjoint operator in $L^2((0,1), a_\beta(t)\,dt)$ associated with (2.2) and for $0 < t < 1$, let

$$u_\beta(t) = -\int_t^1 a_\beta(\tau)^{-1} d\tau. \tag{2.4}$$

We note that for $N = 2$

$$a_\beta(t) \equiv 1, \qquad H_\beta = -\frac{d^2}{dt^2}, \qquad \lambda_\beta = \frac{\pi^2}{4}, \qquad \psi_\beta(t) = \cos\left(\frac{\pi t}{2}\right)$$

for all $\beta \in (0, \pi/2)$. In particular, H_β is independent of β.

For the rest of this section, c and c_i, $i = 1, 2, 3, \dots,$ will denote constants greater than 1 and may depend only on N.

LEMMA 2.1.

(i) *For all $f \in L^2((0,1), a_\beta(t)\,dt)$ we have*

$$\left(H_\beta^{-1} f\right)(t) = \int_{1/2}^{t} a_\beta(\tau) u_\beta(\tau) f(\tau)\,d\tau - u_\beta(t) \int_0^t a_\beta(\tau) f(\tau)\,d\tau$$
$$- \int_{1/2}^{1} a_\beta(\tau) u_\beta(\tau) f(\tau)\,d\tau. \tag{2.5}$$

(ii) *There exists $c \geq 1$ such that if $\beta_1, \beta_2 \in (0, \pi/2)$, then*

$$\|H_{\beta_1}^{-1} f - H_{\beta_2}^{-1} f\|_\infty \leq c|\beta_1 - \beta_2| \|f\|_\infty, \qquad f \in L^\infty(0,1). \tag{2.6}$$

Proof.

(i) For $f \in L^\infty(0,1) \subseteq L^2((0,1), a_\beta(t)\,dt)$, (2.5) can be checked by differentiating the right side. The result now follows since the right side of (2.5) defines a bounded operator on $L^2((0,1), a_\beta(t)\,dt)$ and $L^\infty(0,1)$ is dense in $L^2((0,1), a_\beta(t)\,dt)$.

(ii) Inequality (2.6) is a consequence of (2.4), (2.5), and the estimate

$$\begin{aligned}
\left|a_{\beta_1}(t) - a_{\beta_2}(t)\right| &= \left|\beta_1^{2-N} \sin^{N-2}(\beta_1 t) - \beta_2^{2-N} \sin^{N-2}(\beta_2 t)\right| \\
&\leq c t^{N-2} \left| \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} \beta_1^{2i} t^{2i} - \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} \beta_2^{2i} t^{2i} \right| \\
&\leq c t^{N-2} \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i+1)!} |\beta_1^{2i} - \beta_2^{2i}| \\
&\leq c t^{N-2} \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i)!} \left(\frac{\pi}{2}\right)^{2i} |\beta_1 - \beta_2| \\
&\leq c t^N |\beta_1 - \beta_2|. \quad \blacksquare
\end{aligned}$$

We now wish to show that H_β is ultracontractive with norm independent of β:

LEMMA 2.2. *There exists $c \geq 1$ such that*

$$\|e^{-H_\beta t}\|_{L^2((0,1),\, a_\beta(t)\,dt) \to L^\infty} \leq c t^{-\max\{1,(N-1)/4\}}, \qquad 0 < t < 1 \qquad (2.7)$$

for all $0 < \beta < \pi/2$.

The proof requires special care since we are not working directly with Dirichlet boundary conditions; the natural form core for H_β is

$$\left\{ f \in C^\infty(0,1) \colon f'(0) = f(1) = 0,\ \int_0^1 a_\beta(t) f'(t)^2\,dt < \infty \right\}.$$

Nonetheless, for $N \geq 4$ it is easy to check that $C_c^\infty(0,1)$ is dense in this natural form core and thus we may apply the following result of Davies that applies to self-adjoint operators corresponding to the closures of the quadratic forms of type

$$Q(f) = \int_0^1 a(t) f'(t)^2\,dt, \qquad f \in C_c^\infty(0,1).$$

LEMMA 2.3 [9]. *Suppose ν_i, $i = 1,2,3$, are positive continuous functions on $(0,1)$, let*

$$\delta(t) = \min(t, 1-t), \qquad 0 < t < 1,$$

be the distance to the boundary, and let

$$c = \nu_2^{1/2} \min\left[\left(\nu_1 \nu_2^{-1}\right)^{1/4}, \left(\nu_3 \delta^2\right)^{1/4} \right].$$

Moreover, let

$$H = -\nu_2^{-1} \frac{d}{dt}\left(\nu_1 \frac{d}{dt} \right)$$

be the elliptic operator defined on $L^2((0,1), \nu_2(t)\,dt)$ with Dirichlet boundary conditions. Suppose the following conditions hold:

(i) *There exist $k \geq 1$ and $\mu_i, \epsilon_i \in \mathbf{R}$, $i = 1,2,3$, such that*

$$\begin{cases} k^{-1}\delta(t)^{\mu_i} \leq \nu_i(t) \leq k\,\delta(t)^{\mu_i}, & 0 < t \leq \frac{1}{2} \\ k^{-1}\delta(t)^{\epsilon_i} \leq \nu_i(t) \leq k\,\delta(t)^{\epsilon_i}, & \frac{1}{2} \leq t < 1 \end{cases}$$

for $i = 1,2,3$.

(ii) *There exist* $k_1, k_2 > 0$ *such that* $c^{-k_1} \le k_2(\nu_3 + 1)$.

(iii) *We have* $\nu_3 \le H + 1$ *in the sense of quadratic forms.*

Then there exists $M > 0$ *such that*

$$\|e^{-Ht} f\|_\infty \le M t^{-\mu/4} \|f\|_2, \qquad f \in L^2\big((0,1), \nu_2(t)\, dt\big)$$

for all $0 < t < 1$, *where* $\mu = 1 + 4/k_1$.

Proof of Lemma 2.2. The result is trivially true for $N = 2$ since H_β is independent of β in this case.

Now assume $N = 3$. Then, for all $\beta \in (0, \pi/2)$ and $f \in L^2((0,1), a_\beta(t)\, dt)$,

$$\begin{aligned}
\left|H_\beta^{-1} f(t)\right| &\le \int_{1/2}^t \left|u_\beta(\tau) a_\beta(\tau) f(\tau)\right| d\tau + \int_0^t \left|u_\beta(t) a_\beta(\tau) f(\tau)\right| d\tau \\
&\quad + \int_{1/2}^1 \left|a_\beta(\tau) u_\beta(\tau) f(\tau)\right| d\tau \\
&\le \left\{ \left[\int_{1/2}^t \left|u_\beta(\tau)\right|^2 a_\beta(\tau)\, d\tau\right]^{1/2} + \left|u_\beta(t)\right| \left[\int_0^t a_\beta(\tau)\, d\tau\right]^{1/2} \right. \\
&\quad \left. + \left[\int_{1/2}^1 \left|u_\beta(\tau)\right|^2 a_\beta(\tau)\, d\tau\right]^{1/2} \right\} \|f\|_2 \\
&\le \left\{ \left[\int_{1/2}^t \tau (\ln \tau)^2\, d\tau\right]^{1/2} - t \ln t + c \right\} \|f\|_2 \\
&\le c \|f\|_2 .
\end{aligned}$$

Thus

$$\|H_\beta^{-1} f\|_\infty \le c \|f\|_2, \qquad f \in L^2\big((0,1), a_\beta(t)\, dt\big).$$

So the theory of ultracontractive semigroups (see [8, Theorem 2.4.1]) implies that

$$\|e^{-H_\beta t}\|_{L^2((0,1),\, a_\beta(t)\, dt) \to L^\infty} \le c t^{-1}, \qquad 0 < t < 1. \tag{2.8}$$

For the case $N \ge 4$ we will apply Lemma 2.3 with $\nu_1 = \nu_2 = a_\beta$ and $\nu_3 = c_2^{-1} \delta^{-2}$, where c_2 is determined as follows. Let $\alpha\colon (0,1) \to (0, \infty)$ be a C^∞-function such that

$$\alpha(t) = \begin{cases} (1-t)^{1/2}, & 3/4 \le t < 1 \\ t^{(3-N)/2}, & 0 < t \le 1/4. \end{cases}$$

Let

$$V(t) = -\alpha(t)^{-1} t^{2-N} \frac{d}{dt}\left(t^{N-2} \frac{d\alpha}{dt}\right), \qquad 0 < t < 1$$

so that

$$V(t) \geq \begin{cases} \dfrac{1}{4}(1-t)^{-2}, & 3/4 \leq t < 1 \\ \left(\dfrac{N-3}{2}\right)^2 t^{-2}, & 0 < t \leq 1/4. \end{cases}$$

Thus, by the Allegretto-Piepenbrink theory (see, for example, [9, Proposition 7]), there exists $c_2 \geq 1$ such that

$$\delta^2 \leq c_2\left(H_\beta + 1\right) \tag{2.9}$$

in the sense of quadratic forms. Lemma 2.3(iii) is therefore satisfied.

By (2.3), hypothesis (i) of Lemma 2.3 is satisfied with $\mu_1 = \mu_2 = N - 2$, $\epsilon_1 = \epsilon_2 = 0$, $\mu_3 = \epsilon_3 = -2$. Noting that $c = a_\beta^{1/2}$, (2.3) also shows hypothesis (ii) is satisfied with $k_1 = 4/(N-2)$. Thus we may conclude

$$\|e^{-H_\beta t} f\|_\infty \leq c t^{-(N-1)/4} \|f\|_2, \qquad f \in L^2\left((0,1), a_\beta(t)\, dt\right)$$

for all $t \in (0,1)$. ∎

LEMMA 2.4. *There exists* $c \geq 1$ *such that if* $\beta_1, \beta_2 \in (0, \pi/2)$ *and* $|\beta_1 - \beta_2| \leq 1$, *then*

$$\|\psi_{\beta_1} - \psi_{\beta_2}\|_\infty \leq c|\beta_1 - \beta_2|. \tag{2.10}$$

Proof. Assume $\beta_1 < \beta_2$ and let f and g be positive multiples of ψ_{β_1} and ψ_{β_2}, respectively, such that $\|f\|_{L^2((0,1), a_{\beta_1} dt)} = \|g\|_{L^2((0,1), a_{\beta_2} dt)} = 1$. Let the higher eigenfunctions of H_{β_2} be denoted by $g_2, g_3, \ldots$ and normalized by

$$\|g_i\|_2 = 1, \qquad i = 2, 3, \ldots .$$

Let

$$f = \sigma_1 g + \sum_{i=2}^{\infty} \sigma_i g_i = \sigma_1 g + \tilde{g}. \tag{2.11}$$

We have, by (2.6), Lemma 2.2, and the estimate in the proof of Lemma 2.1(ii),

$$\begin{aligned}
\left|\langle H_{\beta_2}^{-1}f, f\rangle_{\beta_2} - \lambda_{\beta_1}^{-1}\right| &= \left|\langle H_{\beta_2}^{-1}f, f\rangle_{\beta_2} - \langle H_{\beta_1}^{-1}f, f\rangle_{\beta_1}\right| \\
&= \Bigg|\int_0^1 \left[\left(H_{\beta_2}^{-1}f\right)(t) - \left(H_{\beta_1}^{-1}f\right)(t)\right] f(t)\, a_{\beta_2}(t)\, dt \\
&\qquad + \int_0^1 \left(H_{\beta_1}^{-1}f\right)(t) f(t)\left[a_{\beta_2}(t) - a_{\beta_1}(t)\right] dt\Bigg| \\
&\le c\|H_{\beta_2}^{-1} - H_{\beta_1}^{-1}\|_{L^\infty \to L^\infty}\|e^{\lambda_{\beta_1}}e^{-H_{\beta_1}}f\|_\infty\|f\|_1 \\
&\qquad + c\|f\|_\infty\|f\|_1|\beta_1 - \beta_2| \\
&\le c|\beta_1 - \beta_2|e^{\lambda_{\beta_1}}\|f\|_2\|f\|_1 + c\|f\|_\infty\|f\|_1|\beta_1 - \beta_2| \\
&\le c|\beta_1 - \beta_2|\,\|f\|_2^2 \\
&\le c|\beta_1 - \beta_2|.
\end{aligned}$$

From the calculation above and minimax we have

$$\lambda_{\beta_2}^{-1} \le \langle H_{\beta_2}^{-1}f, f\rangle \le \lambda_{\beta_1}^{-1} + c|\beta_2 - \beta_1|.$$

Hence

$$\begin{aligned}
|\lambda_{\beta_1} - \lambda_{\beta_2}| &\le c\lambda_{\beta_1}\lambda_{\beta_2}|\beta_1 - \beta_2| \\
&\le c|\beta_1 - \beta_2|, \qquad (2.12)
\end{aligned}$$

since $\lambda_\beta = \beta^2 E_\beta$ tends to a constant depending only on N as $\beta \to 0$ (see [11]).

Next we have for all $k \in \mathbf{N}$,

$$\begin{aligned}
(\sigma_1 - 1)g + \tilde{g} &= \left(\lambda_{\beta_1}^k - \lambda_{\beta_2}^k\right)H_{\beta_1}^{-k}f + \lambda_{\beta_2}^k\left(H_{\beta_1}^{-k} - H_{\beta_2}^{-k}\right)f \\
&\qquad + \lambda_{\beta_2}^k H_{\beta_2}^{-k}\left[(\sigma_1 - 1)g + \tilde{g}\right],
\end{aligned}$$

and thus

$$\tilde{g} - \lambda_{\beta_2}^k H_{\beta_2}^{-k}\tilde{g} = \left(\lambda_{\beta_1}^k - \lambda_{\beta_2}^k\right)H_{\beta_1}^{-k}f + \lambda_{\beta_2}^k\left(H_{\beta_1}^{-k} - H_{\beta_2}^{-k}\right)f. \qquad (2.13)$$

By (2.12), Lemma 2.1(ii), Lemma 2.2, and (2.13) we have

$$\begin{aligned} c|\beta_1 - \beta_2|^2 &\geq \|\tilde{g} - \lambda_{\beta_2}^k H_{\beta_2}^{-k} \tilde{g}\|_\infty^2 \\ &\geq c^{-1} \|\tilde{g} - \lambda_{\beta_2}^k H_{\beta_2}^{-k} \tilde{g}\|_2^2 \\ &\geq c^{-1} \left\| \sum_{n=2}^{\infty} \left(1 - \lambda_{\beta_2}^k E_n^{-k}\right) \sigma_n g_n \right\|_2^2 \\ &\geq c^{-1} \sum_{n=2}^{\infty} \sigma_n^2 \\ &= c^{-1} \|\tilde{g}\|_2^2. \end{aligned} \tag{2.14}$$

By Lemma 2.2 and [8, Theorem 2.4.1] there exists $c \geq 1$ such that

$$\|H_\beta^{-2^N}\|_{L^2((0,1),\, a_\beta(t)\, dt) \to L^\infty} \leq c$$

for all $\beta \in (0, \pi/2)$. Therefore, by (2.6), (2.7), (2.12)–(2.14), we have

$$\begin{aligned} \|\tilde{g}\|_\infty &\leq |\lambda_{\beta_1}^{2^N} - \lambda_{\beta_2}^{2^N}| e^{\lambda_{\beta_1}} \|e^{-H_{\beta_1}} H_{\beta_1}^{-2^N} f\|_\infty \\ &\quad + \lambda_{\beta_2}^{2^N} \|H_{\beta_1}^{-2^N} - H_{\beta_2}^{-2^N}\|_{L^\infty \to L^\infty} e^{\lambda_{\beta_1}} \|e^{-H_{\beta_1}} f\|_\infty \\ &\quad + \lambda_{\beta_2}^k \|H_{\beta_2}^{-2^N}\|_{L^2 \to L^\infty} \|\tilde{g}\|_2 \\ &\leq c|\beta_1 - \beta_2|. \end{aligned} \tag{2.15}$$

Since $\|f\|_2 = 1$, (2.11) and (2.15) imply that

$$\|f - g\|_\infty \leq c|\beta_1 - \beta_2|,$$

which implies (2.10). ∎

COROLLARY 2.

(i) *There exists $c \geq 1$ such that if $\beta_1, \beta_2 \in (0, \pi/2)$, then*

$$\left|\psi_{\beta_1}(t) - \psi_{\beta_2}(t)\right| \leq c|\beta_1 - \beta_2|(1 - t), \qquad \tfrac{1}{2} \leq t < 1. \tag{2.16}$$

(ii) *There exists $c \geq 1$ such that*

$$\psi_\beta(t) \geq c^{-1}(1 - t), \qquad \tfrac{1}{2} \leq t < 1 \tag{2.17}$$

for all $\beta \in (0, \pi/2)$.

Proof. From (2.2) we have

$$\psi_\beta(t) = \int_t^1 \int_0^\tau \lambda_\beta a_\beta(\tau)^{-1} a_\beta(s) \psi_\beta(s)\, ds\, d\tau. \tag{2.18}$$

Inequality (2.16) now follows from Lemma 2.4, (2.12), and (2.18).
To prove (2.17) we note that

$$\left|\psi_\beta'(t)\right| = \lambda_\beta a_\beta(t)^{-1} \int_0^t a_\beta(\tau) \psi_\beta(\tau)\, d\tau \le ct.$$

Thus if $\epsilon > 0$ is sufficiently small, then

$$\psi_\beta(t) \ge \tfrac{1}{2}, \qquad 0 < t \le \epsilon.$$

Hence for $\frac{1}{2} \le t < 1$ we have

$$\begin{aligned}\left|\psi_\beta'(t)\right| &= \lambda_\beta a_\beta(t)^{-1} \int_0^t a_\beta(\tau) \psi_\beta(\tau)\, d\tau \\ &\ge c^{-1} \int_0^\epsilon \tau^{N-2}\, d\tau \\ &= c^{-1}\end{aligned}$$

which implies (2.17) since $\psi_\beta(1) = 0$. ∎

Remark. We note that a modification of the proof of (2.12) will give

$$|\lambda_\beta - \gamma| \le c\beta^2, \qquad 0 < \beta < \pi/2 \tag{2.19}$$

for some $c \ge 1$, where γ is the ground state eigenvalue of the elliptic operator

$$H\psi = -t^{2-N} \frac{d}{dt}\left(t^{N-2} \frac{d\psi}{dt}\right)$$

defined in $L^2((0,1), t^{N-2}\, dt)$ with boundary conditions

$$\psi'(0) = \psi(1) = 0$$

(i.e., γ is the ground state eigenvalue of the unit ball in $\mathbf{R}^{N-1}$).

LEMMA 2.5. *Let $R > 0$ and $\beta \in (0, \pi/4)$. Then there exist $c_3, c_4 \geq 1$ such that*

$$\begin{aligned}\Bigg| \theta^{-1} \sin^{-1}&\left\{ \frac{R \sin \theta}{\left[R^2 + \delta^2 - 2R\delta \cos \theta\right]^{1/2}} \right\} \\ &- \beta^{-1} \sin^{-1}\left\{ \frac{R \sin \beta}{\left[R^2 + \delta^2 - 2R\delta \cos \beta\right]^{1/2}} \right\} \Bigg| \\ &\leq c_4 R^{-1} \delta(\beta - \theta) \end{aligned} \tag{2.20}$$

for all $0 \leq \theta \leq \beta$ and $0 < \delta < c_3^{-1} R$.

Proof. By replacing δ/R with δ, one sees that it suffices to prove (2.20) for $R = 1$. Let

$$\alpha(\theta) = \sin^{-1}\left\{ \frac{\sin \theta}{\left[1 + \delta^2 - 2\delta \cos \theta\right]^{1/2}} \right\}, \qquad \theta \in (0, \beta).$$

Then $\alpha(\theta)$ can be rewritten as

$$\alpha(\theta) = \cot^{-1}\left\{ \frac{\cos \theta - \delta}{\sin \theta} \right\}.$$

Therefore

$$\alpha'(\theta) = \frac{1 - \delta \cos \theta}{1 + \delta^2 - 2\delta \cos \theta}$$

and

$$\alpha''(\theta) = \frac{\delta(\delta^2 - 1)\sin \theta}{\left(1 + \delta^2 - 2\delta \cos \theta\right)^2}. \tag{2.21}$$

For some $\zeta, \eta \in [0, \theta]$ we have

$$\begin{aligned}\left(\frac{\alpha(\theta)}{\theta} \right)' &= \frac{\theta \alpha'(\theta) - \alpha(\theta)}{\theta^2} \\ &= \theta^{-2}\left\{ \theta\left[\alpha'(0) + \alpha''(\eta)\theta\right] - \left[\alpha'(0)\theta + \tfrac{1}{2}\alpha''(\zeta)\theta^2\right] \right\} \\ &= \alpha''(\eta) - \tfrac{1}{2}\alpha''(\zeta). \end{aligned}$$

Hence, by (2.21), there exist $c_3, c_4 \geq 1$ such that

$$\left| \left(\frac{\alpha(\theta)}{\theta} \right)' \right| \leq c_4 \delta, \qquad \theta \in (0, \beta), \qquad \delta \in \left(0, c_3^{-1}\right).$$

The lemma now follows from the Mean Value Theorem. ∎

We next consider two cones with the same axis of symmetry and whose cross-sections are shown in Fig. 1. The cone on the left of Fig. 1 will be denoted by $K(i)$ and the one on the right by $K(i+1)$. We shall use the notation

$$\left.\begin{aligned} R_j &= A_j - B_j \\ -\mu_j &= \text{slope of the segment } P_j A_j \\ S_j &= \text{spherical part of } \partial K(j) \end{aligned}\right\}, \qquad j = i, i+1.$$

It can be checked that for any $z \in S_{i+1}$,

$$\begin{aligned} \theta_i &= \theta_i(z) \\ &= \sin^{-1}\left\{ \frac{R_{i+1} \sin \theta_{i+1}}{\left[R_{i+1}^2 + (A_{i+1} - A_i)^2 - 2(A_{i+1} - A_i) R_{i+1} \cos \theta_{i+1}\right]^{1/2}} \right\}. \end{aligned} \tag{2.22}$$

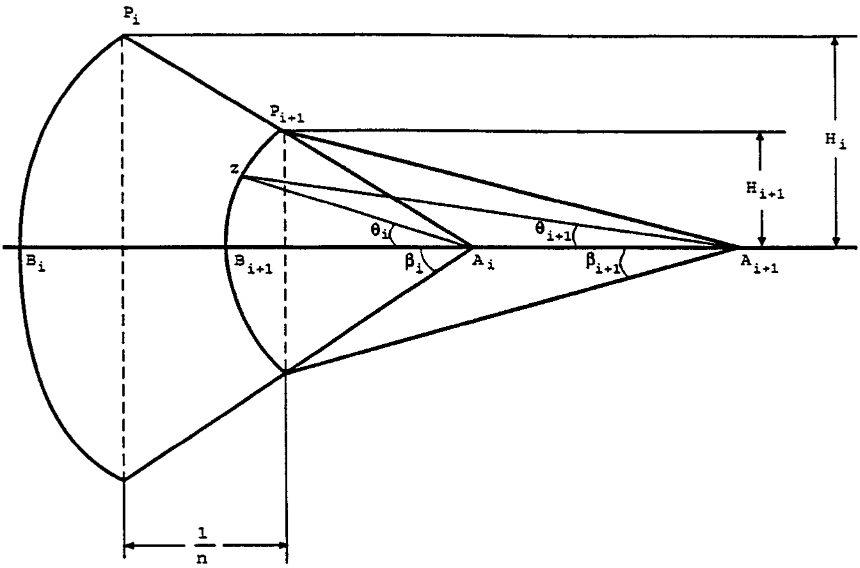

FIGURE 1

Let h_j be the positive rotationally symmetric function defined on $K(j)$, $j = i, i+1$, by

$$h_j(z) = \left(\frac{|z - A_j|}{R_j} \right)^{\alpha(\beta_j)} \phi_{\beta_j}(\theta_j), \qquad z \in K(j), \tag{2.23}$$

where

$$\alpha(\beta_j) = \frac{2 - N + \sqrt{(N-2)^2 + 4E_{\beta_j}}}{2}. \tag{2.24}$$

Observe that h_j is harmonic by our choice of $\alpha(\beta_j)$. Also note that $h_j \leq \phi_{\beta_j}$ on K_j, $h_j = \phi_{\beta_j}$ on S_j, and that $h_j = 0$ on $\partial K_j \setminus S_j$.

Since we are interested in a lower bound for $\phi(x, 0)$ when x is large, we must study the asymptotic behavior of various quantities as $R_j \to \infty$, and $\mu_j, \beta_j, \theta_j \downarrow 0$. We shall use the "big O" notation in the following sense: for two expressions F and G involving H_i, R_i, μ_i, n, etc., $G = O(F)$ means that there exists $c \geq 1$ depending only on the dimension N such that $G \leq cF$ provided that for some integer $n_o(x, \rho), n \geq n_o$.

LEMMA 2.6. *Let h_i, h_{i+1} be the rotationally symmetric harmonic functions defined by* (2.23). *Then there exists $c \geq 1$ such that whenever $\beta_i \leq \pi/4$*

$$h_i(z) \geq \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \right)^{\alpha(\beta_i)} (1 - c(\beta_i - \beta_{i+1})) h_{i+1}(z)$$

for all $z \in S_{i+1}$.

Proof. First note that (2.2) and (2.3) imply that there exists $c \geq 1$ such that

$$|\psi_\beta'(t)| \leq ct, \qquad t \in (0,1), \qquad \beta \in (0, \pi/2). \tag{2.25}$$

Applying Lemma 2.6 with $\beta = \beta_{i+1}$, $\theta = \theta_{i+1}$, $\delta = A_{i+1} - A_i$, $R = R_{i+1}$ and using (2.22), we get

$$\frac{\beta_i}{\theta_{i+1}} \left| \frac{\theta_i}{\beta_i} - \frac{\theta_{i+1}}{\beta_{i+1}} \right| = \left| \frac{\theta_i}{\theta_{i+1}} - \frac{\beta_i}{\beta_{i+1}} \right| \leq c_4 (A_{i+1} - A_i) R_{i+1}^{-1} (\beta_{i+1} - \theta_{i+1}).$$

Since $\theta_{i+1}/\beta_i \leq 1$,

$$\frac{\theta_i}{\beta_i} = \frac{\theta_{i+1}}{\beta_{i+1}} + O\left(\frac{(A_{i+1} - A_i)(\beta_{i+1} - \theta_{i+1})}{R_{i+1}} \right), \qquad z \in S_{i+1}.$$

Hence, by (2.25), (2.10), and (2.16), we have

$$
\begin{aligned}
\phi_{\beta_i}(\theta_i) &= \psi_{\beta_i}\left(\frac{\theta_i}{\beta_i}\right) = \psi_{\beta_i}\left(\frac{\theta_{i+1}}{\beta_{i+1}} + O\left(\frac{(A_{i+1} - A_i)(\beta_{i+1} - \theta_{i+1})}{R_{i+1}}\right)\right) \\
&= \psi_{\beta_i}\left(\frac{\theta_{i+1}}{\beta_{i+1}}\right) + O\left(\frac{(A_{i+1} - A_i)(\beta_{i+1} - \theta_{i+1})}{R_{i+1}}\right) \\
&= \psi_{\beta_{i+1}}\left(\frac{\theta_{i+1}}{\beta_{i+1}}\right) + O\left((\beta_i - \beta_{i+1})\left(1 - \frac{\theta_{i+1}}{\beta_{i+1}}\right)\right) \\
&\quad + O\left(\frac{(A_{i+1} - A_i)(\beta_{i+1} - \theta_{i+1})}{R_{i+1}}\right)
\end{aligned}
$$

for all $z \in S_{i+1}$. Therefore, by (2.17),

$$
\phi_{\beta_i}(\theta_i)\,\phi_{\beta_{i+1}}(\theta_{i+1})^{-1} = 1 + O(\beta_i - \beta_{i+1}) + O\left(\frac{(A_{i+1} - A_i)\,\beta_{i+1}}{R_{i+1}}\right). \tag{2.26}
$$

We next estimate the term $O((A_i - A_{i+1})\beta_{i+1}/R_{i+1})$ in (2.26). Using the equations

$$
\begin{gathered}
\sin(\beta_j) = H_j R_j^{-1} \\
R_j = H_j\,\mu_j^{-1}\sqrt{1 + \mu_j^2} \\
A_{i+1} - A_i = H_{i+1}\,\mu_{i+1}^{-1} - H_i\,\mu_i^{-1} + n^{-1}, \qquad j = i, i+1 \\
H_{i+1} = H_i - \mu_i n^{-1}
\end{gathered} \tag{2.27}
$$

we have

$$
\begin{aligned}
\frac{\beta_{i+1}(A_{i+1} - A_i)}{R_{i+1}} &= \frac{\mu_{i+1}}{H_{i+1}\sqrt{1 + \mu_{i+1}^2}}\,\sin^{-1}\left(\frac{H_{i+1}}{R_{i+1}}\right)\left[\frac{H_{i+1}}{\mu_{i+1}} - \frac{H_i}{\mu_i} + \frac{1}{n}\right] \\
&= \frac{\mu_{i+1}}{(H_i - \mu_i/n)\sqrt{1 + \mu_{i+1}^2}}\,\sin^{-1}\left(\frac{\mu_{i+1}}{\sqrt{1 + \mu_{i+1}^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\frac{H_i}{\mu_{i+1}} - \frac{H_i}{\mu_i} + \frac{1}{n}\left(1 - \frac{\mu_i}{\mu_{i+1}}\right)\right] \\
&= \frac{1}{\mu_{i+1}}\left(\frac{\mu_i}{\mu_{i+1}} - 1\right)\frac{\mu_{i+1}}{\sqrt{1+\mu_{i+1}^2}} \sin^{-1}\left(\frac{\mu_{i+1}}{\sqrt{1+\mu_{i+1}^2}}\right) \\
&= \frac{1}{\mu_{i+1}^2}(\mu_i - \mu_{i+1}) O(\mu_{i+1}^2) \\
&= O(\mu_i - \mu_{i+1}) \\
&= O(\beta_i - \beta_{i+1}).
\end{aligned}
$$

Thus

$$\phi_{\beta_i}(\theta_i)\,\phi_{\beta_{i+1}}(\theta_{i+1})^{-1} = 1 + O(\beta_i - \beta_{i+1}).$$

Finally, since $\mu_{i+1} < \mu_i$, for all $z \in S_{i+1}$, we have

$$|z - A_i| \geq A_i - B_{i+1} = R_{i+1} - A_{i+1} + A_i.$$

Thus, if $z \in S_{i+1}$, then for some $c \geq 1$

$$
\begin{aligned}
h_i(z) &= \left(\frac{|z - A_i|}{R_i}\right)^{\alpha(\beta_i)} \phi_{\beta_i}(\theta_i) \\
&\geq \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i}\right)^{\alpha(\beta_i)} \left[\phi_{\beta_i}(\theta_i)\,\phi_{\beta_{i+1}}(\theta_{i+1})^{-1}\right] \phi_{\beta_{i+1}}(\theta_{i+1}) \\
&\geq \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i}\right)^{\alpha(\beta_i)} \left[1 - c(\beta_i - \beta_{i+1})\right] h_{i+1}(z)
\end{aligned}
$$

which implies the lemma. ∎

LEMMA 2.7. *We have*

$$
\begin{aligned}
&\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \\
&\qquad = 1 - \frac{1}{nR_i}\sqrt{1+\mu_i^2} + O\left(\frac{H_i}{R_i}(\beta_i - \beta_{i+1})\right) + O\left(\frac{1}{n}(\beta_i - \beta_{i+1})\right).
\end{aligned}
$$

Proof. The lemma follows from the following calculations using (2.27):

$$
\begin{aligned}
&R_{i+1} - A_{i+1} + A_i \\
&\quad = \frac{1}{\mu_{i+1}}\left(H_i - \frac{\mu_i}{n}\right)\sqrt{1+\mu_{i+1}^2} - \frac{1}{n} - \left(H_i - \frac{\mu_i}{n}\right)\frac{1}{\mu_{i+1}} + \frac{H_i}{\mu_i} \\
&\quad = H_i\left(\frac{1}{\mu_i} - \frac{1}{\mu_{i+1}}\right) + \frac{1}{n}\left(\frac{\mu_i}{\mu_{i+1}} - 1 - \frac{\mu_i}{\mu_{i+1}}\sqrt{1+\mu_{i+1}^2}\right) \\
&\qquad + \frac{H_i}{\mu_{i+1}}\sqrt{1+\mu_{i+1}^2} \\
&\quad = R_i - \frac{1}{n}\frac{1}{\mu_{i+1}}\left[\mu_i\left(1 + \frac{1}{2}\mu_{i+1}^2 - \frac{1}{8}\mu_{i+1}^4 + \cdots\right) + \mu_{i+1} - \mu_i\right] \\
&\qquad + H_i\left(\frac{1}{\mu_i} - \frac{1}{\mu_{i+1}}\right) + H_i\left(\frac{1}{\mu_{i+1}}\sqrt{1+\mu_{i+1}^2} - \frac{1}{\mu_i}\sqrt{1+\mu_i^2}\right) \\
&\quad = R_i - \frac{1}{n\mu_{i+1}}\left[\mu_{i+1}\sqrt{1+\mu_{i+1}^2}\right. \\
&\qquad\qquad \left. + (\mu_i - \mu_{i+1})\left(\frac{1}{2}\mu_{i+1}^2 - \frac{1}{8}\mu_{i+1}^4 + \cdots\right)\right] \\
&\qquad + H_i\left[\frac{1}{2}(\mu_{i+1} - \mu_i) - \frac{1}{8}(\mu_{i+1}^3 - \mu_i^3) + \cdots\right] \\
&\quad = R_i - \frac{1}{n}\sqrt{1+\mu_i^2} + \frac{1}{n}\left(\sqrt{1+\mu_i^2} - \sqrt{1+\mu_{i+1}^2}\right) \\
&\qquad + O\left(\frac{1}{n}(\mu_i - \mu_{i+1})\right) \\
&\qquad + O(H_i(\mu_i - \mu_{i+1})). \qquad \blacksquare
\end{aligned}
$$

3. PROOF OF THEOREM 1

In this section $\rho\colon [0,\infty) \to (0,\infty)$ will be a continuous function satisfying (A1) and (A2), and D_ρ will be the horn-shaped region defined by (1.1). We shall let $x > 1$ and n be a large positive integer. Referring to Fig. 2, we construct a finite sequence of cones $K(i)$, $i = 1, \ldots, [nx]$, inside D_ρ with $\mu_i = -\rho'(1 + in^{-1})$. The symbols c and c_i, $i = 1, 2, 3, \ldots,$ will denote constants ≥ 1 which depend only on N and ρ. We shall let $F(H_i, R_i, \mu_i)$

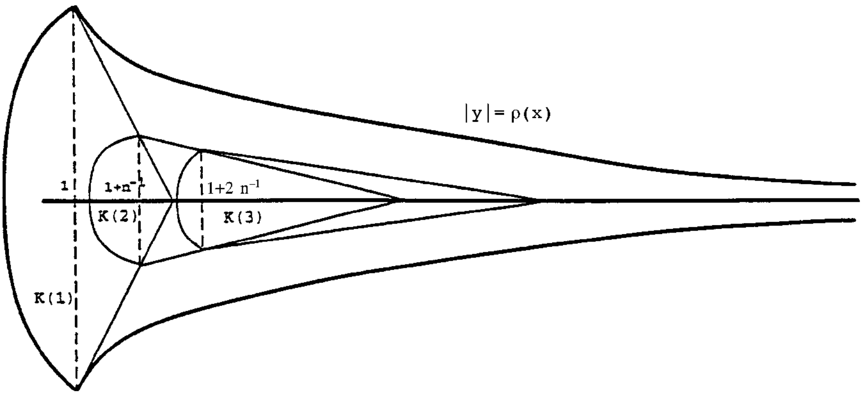

FIGURE 2

represent a function of H_i, R_i, μ_i which may change at each occurrence. Note that for each function F there exists a constant $C = C(x, \rho)$ such that $F(H_i, R_i, \mu_i) \leq C$ since the quantities involved depend only on the function ρ restricted to the interval $[1, x]$.

LEMMA 3.1.

(i) *If $N = 2$, then there exists $c \geq 1$ such that for all $x \geq 1$,*

$$\lim_{n\to\infty} \sum_{i=1}^{[nx]} \ln\left[\left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i}\right)^{\alpha(\beta_i)}\right]$$
$$\geq -\frac{\pi}{2}\left\{\int_1^x \frac{dt}{\rho(t)} + \frac{1}{3}\int_1^x \frac{\rho'(t)^2}{\rho(t)}\,dt\right.$$
$$\left. - \frac{1}{24}\int_1^x \frac{\rho'(t)^4}{\rho(t)}\,dt + O\left(\int_1^x \frac{\rho'(t)^8}{\rho(t)}\,dt\right)\right\} - c. \quad \text{(3.1a)}$$

(ii) *If $N \geq 3$, then there exists $c \geq 1$ such that for all $x \geq 1$,*

$$\lim_{n\to\infty} \sum_{i=1}^{[nx]} \ln\left[\left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i}\right)\right]^{\alpha(\beta_i)}$$
$$\geq -\sqrt{\gamma}\int_1^x \frac{dt}{\rho(t)} + \left(\frac{N}{2} - 1\right)\int_1^x \frac{|\rho'(t)|}{\rho(t)}\,dt$$
$$+ O\left(\int_1^x \frac{\rho'(t)^2}{\rho(t)}\,dt\right) - c. \quad \text{(3.1b)}$$

Proof. For $N = 2$ we have

$$\alpha(\beta_i) = \frac{\pi}{2\beta_i} = \frac{\sqrt{\gamma}}{\beta_i} \tag{3.2a}$$

and for $N \geq 3$, (2.19), (2.24), (2.28) imply that for all sufficiently small β_i,

$$\begin{aligned}
\alpha(\beta_i) &= \frac{1}{2}\left\{2 - N + 2\sqrt{E_{\beta_i}}\left[1 + \frac{1}{2}\left(\frac{(N-2)^2}{4E_{\beta_i}}\right)\right.\right. \\
&\qquad \left.\left. - \frac{1}{8}\left(\frac{(N-2)^4}{16E_{\beta_i}^2}\right) + \cdots\right]\right\} \\
&= \sqrt{E_{\beta_i}} + \left(1 - \frac{N}{2}\right) + O\left(E_{\beta_i}^{-1/2}\right) \\
&= \frac{\sqrt{\gamma}}{\beta_i}\sqrt{1 + O(\beta_i^2)} + \left(1 - \frac{N}{2}\right) + O(\beta_i) \\
&= \frac{\sqrt{\gamma}}{\beta_i} + O(\beta_i) + \left(1 - \frac{N}{2}\right).
\end{aligned} \tag{3.2b}$$

For all $N \geq 2$ the leading term is estimated as follows: by Lemma 2.8 we have, for all large n,

$$\begin{aligned}
&\sqrt{\gamma}\sum_{i=1}^{[nx]} \frac{1}{\beta_i} \ln\left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i}\right) \\
&\quad = \sqrt{\gamma}\sum_{i=1}^{[nx]}\left[\sin^{-1}\left(\frac{H_i}{R_i}\right)\right]^{-1} \ln\left[1 - \frac{1}{nR_i}\sqrt{1 + \mu_i^2}\right. \\
&\qquad \left. + O\left(\frac{H_i}{R_i}(\beta_i - \beta_{i+1})\right) + \frac{1}{n}O(\beta_i - \beta_{i+1})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\gamma} \sum_{i=1}^{[nx]} \frac{R_i}{H_i} \left[1 - \frac{1}{6} \left(\frac{H_i}{R_i} \right)^2 - \frac{17}{360} \left(\frac{H_i}{R_i} \right)^4 + \cdots \right] \\
&\quad \times \Bigg\{ - \left[\frac{\sqrt{1+\mu_i^2}}{nR_i} + O\left(\frac{H_i}{R_i} (\beta_i - \beta_{i+1}) \right) + \frac{1}{n} O(\beta_i - \beta_{i+1}) \right] \\
&\qquad + \frac{1}{2} \left[\frac{\sqrt{1+\mu_i^2}}{nR_i} + O\left(\frac{H_i}{R_i} (\beta_i - \beta_{i+1}) \right) + \frac{1}{n} O(\beta_i - \beta_{i+1}) \right]^2 \\
&\qquad - \frac{1}{3} \Bigg[\frac{\sqrt{1+\mu_i^2}}{nR_i} + O\left(\frac{H_i}{R_i} (\beta_i - \beta_{i+1}) \right) \\
&\qquad\qquad + \frac{1}{n} O(\beta_i - \beta_{i+1}) \Bigg]^3 + \cdots \Bigg\} \qquad (3.3) \\
&\geq \sqrt{\gamma} \sum_{i=1}^{[nx]} \Bigg\{ - \frac{\sqrt{1+\mu_i^2}}{nH_i} + O(\beta_i - \beta_{i+1}) \\
&\qquad + \frac{1}{n} O\big(F(H_i, R_i, \mu_i)(\beta_i - \beta_{i+1})\big) \\
&\qquad + \frac{1}{6} \left(\frac{H_i \sqrt{1+\mu_i^2}}{nR_i^2} \right) + \frac{1}{n^2} O\big(F(H_i, R_i, \mu_i)\big) \Bigg\} - c \\
&\geq \sqrt{\gamma} \sum_{i=1}^{[nx]} \Bigg\{ \frac{-1}{nH_i} - \frac{1}{2} \frac{\mu_i^2}{nH_i} + \frac{1}{8} \frac{\mu_i^4}{nH_i} + \frac{1}{6} \frac{\mu_i^2}{nH_i} - \frac{1}{12} \frac{\mu_i^4}{nH_i} \\
&\qquad + O(\beta_i - \beta_{i+1}) + \frac{1}{n} O\big(F(H_i, R_i, \mu_i)(\beta_i - \beta_{i+1})\big) \\
&\qquad + \frac{1}{n^2} O\big(F(H_i, R_i, \mu_i)\big) + O\left(\frac{\mu_i^8}{nH_i} \right) \Bigg\} - c \\
&= \sqrt{\gamma} \sum_{i=1}^{[nx]} \Bigg\{ - \frac{1}{nH_i} - \frac{\mu_i^2}{3nH_i} + \frac{1}{24} \frac{\mu_i^4}{nH_i} + O(\beta_i - \beta_{i+1}) \\
&\qquad + O\left(\frac{1}{n} F(H_i, R_i, \mu_i)(\beta_i - \beta_{i+1}) \right) + O\left(\frac{1}{n^2} F(H_i, R_i, \mu_i) \right) \\
&\qquad + O\left(\frac{\mu_i^8}{nH_i} \right) \Bigg\} - c.
\end{aligned}
$$

We next note that

$$\lim_{n\to\infty}\sqrt{\gamma}\sum_{i=1}^{[nx]}\left\{-\frac{1}{nH_i}-\frac{\mu_i^2}{3nH_i}+\frac{1}{24}\frac{\mu_i^4}{nH_i}\right\}$$
$$=-\sqrt{\gamma}\int_1^x\frac{dt}{\rho(t)}-\frac{\sqrt{\gamma}}{3}\int_1^x\frac{\rho'(t)^2}{\rho(t)}\,dt+\frac{\sqrt{\gamma}}{24}\int_1^x\frac{\rho'(t)^4}{\rho(t)}\,dt, \quad (3.4)$$

$$\lim_{n\to\infty}\sum_{i=1}^{[nx]}O(\beta_i-\beta_{i+1})=O(1), \quad (3.5)$$

$$\lim_{n\to\infty}\sum_{i=1}^{[nx]}\left\{O\left(\frac{1}{n}F(H_i,R_i,\mu_i)(\beta_i-\beta_{i+1})\right)+O\left(\frac{1}{n^2}F(H_i,R_i,\mu_i)\right)\right\}=0, \quad (3.6)$$

$$\lim_{n\to\infty}\sum_{i=1}^{[nx]}\frac{\mu_i^8}{nH_i}=\int_1^x\frac{\rho'(t)^8}{\rho(t)}\,dt. \quad (3.7)$$

For $N\geq 3$ we estimate the remaining terms as

$$\sum_{i=1}^{[nx]}\left\{\left[1-\frac{N}{2}+O(\beta_i)\right]\right.$$
$$\left.\times\ln\left[1-\frac{\sqrt{1+\mu_i^2}}{nR_i}+O\left(\frac{H_i}{R_i}(\beta_i-\beta_{i+1})\right)+\frac{1}{n}O(\beta_i-\beta_{i+1})\right]\right\}$$
$$=\sum_{i=1}^{[nx]}\left[1-\frac{N}{2}+O(\beta_i)\right]$$
$$\times\left[\frac{-\mu_i}{nH_i}+O\left(\frac{H_i}{R_i}(\beta_i-\beta_{i+1})\right)+\frac{1}{n}O(\beta_i-\beta_{i+1})\right]$$
$$\times\left\{1+\frac{1}{2}\left[\frac{-\mu_i}{nH_i}+O\left(\frac{H_i}{R_i}(\beta_i-\beta_{i+1})\right)+\frac{1}{n}O(\beta_i-\beta_{i+1})\right]\right.$$
$$\left.+\frac{1}{3}\left[\frac{-\mu_i}{nH_i}+O\left(\frac{H_i}{R_i}(\beta_i-\beta_{i+1})\right)+\frac{1}{n}O(\beta_i-\beta_{i+1})\right]^2+\cdots\right\}$$

$$= \sum_{i=1}^{[nx]} \left\{ \left(\frac{N}{2} - 1 \right) \frac{\mu_i}{nH_i} + O\left(\frac{\mu_i^2}{nH_i} \right) + O(\beta_i - \beta_{i+1}) \right.$$

$$\left. + O\left(\frac{1}{n^2} F(H_i, R_i, \mu_i) \right) + O\left(\frac{1}{n} F(H_i, R_i, \mu_i)(\beta_i - \beta_{i+1}) \right) \right\} \tag{3.8}$$

and hence

$$\lim_{n \to \infty} \sum_{i=1}^{[nx]} \left\{ \left(1 - \frac{N}{2} + O(\beta_i) \right) \right.$$

$$\left. \times \ln \left[1 - \frac{\sqrt{1 + \mu_i^2}}{nR_i} + O\left(\frac{H_i}{R_i} (\beta_i - \beta_{i+1}) \right) + \frac{1}{n} O(\beta_i - \beta_{i+1}) \right] \right\}$$

$$\geq \left(\frac{N}{2} - 1 \right) \int_1^x \frac{|\rho'(t)|}{\rho(t)} \, dt + O\left(\int_1^x \frac{\rho'(t)^2}{\rho(t)} \, dt \right) - c \tag{3.9}$$

for some $c \geq 1$. The lemma now follows from (3.3) through (3.9). ∎

LEMMA 3.2. *There exists $c_5 \geq 1$ such that*

$$\lim_{n \to \infty} \sum_{i=1}^{[nx]} \ln[1 - c(\beta_i - \beta_{i+1})] \geq -c_5. \tag{3.10}$$

Proof. By (3.5) we have

$$\sum_{i=1}^{[nx]} \ln[1 - c(\beta_i - \beta_{i+1})] = -\sum_{i=1}^{[nx]} O(\beta_i - \beta_{i+1})$$
$$\geq -c_5$$

for all sufficiently large n. ∎

We may now finish the proof of Theorem 1. It should be noticed that this construction of overlapping cones allows us to obtain a lower bound without repeated recourse to the Harnack principle and thus prevents powers of the constant becoming unbounded.

Proof of Theorem 1. Choose $\tilde{c} \geq 1$ so that

$$\phi(z) \geq \tilde{c}^{-1} h_1(z), \qquad \forall z \in K(1).$$

Note that $\tilde{c}$ does not depend on x. Since $S_2 \subset K(1)$ we obviously have

$$\phi(z) \geq \tilde{c}^{-1} h_1(z), \qquad \forall z \in S_2$$

and so Lemma 2.7 implies that

$$\phi(z) \geq \tilde{c}^{-1} \left(\frac{R_2 - A_2 + A_1}{R_1} \right)^{\alpha(\beta_1)} (1 - c(\beta_1 - \beta_2)) h_2(z), \qquad \forall z \in S_2. \tag{3.11}$$

Now because ϕ is superharmonic and that (3.11) holds on $\partial K(2)$, we have

$$\phi(z) \geq \tilde{c}^{-1} \left(\frac{R_2 - A_2 + A_1}{R_1} \right)^{\alpha(\beta_1)} (1 - c(\beta_1 - \beta_2)) h_2(z), \quad \forall z \in K(2).$$

Moreover, since $S_3 \subset K(2)$ we may apply Lemma 2.7 to give

$$\phi(z) \geq \tilde{c}^{-1} h_3(z) \prod_{i=1}^{2} \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \right)^{\alpha(\beta_i)} (1 - c(\beta_i - \beta_{i+1})), \qquad \forall z \in S_3.$$

A simple induction argument shows that

$$\phi(x, 0) \geq \tilde{c}^{-1} h_{[nx]}(x, 0) \prod_{i=1}^{[nx]} \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \right)^{\alpha(\beta_i)} (1 - c(\beta_i - \beta_{i+1})).$$

By our construction of cones and the normalization $\phi_\beta(0) = 1$,

$$\lim_{n \to \infty} h_{[nx]}(x, 0) = \lim_{n \to \infty} \phi_{\beta_{[nx]}}(0) = 1.$$

Thus,

$$\begin{aligned} \phi(x, 0) &\geq \tilde{c}^{-1} \lim_{n \to \infty} \prod_{i=1}^{[nx]} \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \right)^{\alpha(\beta_i)} (1 - c(\beta_i - \beta_{i+1})) \\ &= \tilde{c}^{-1} \exp \left\{ \lim_{n \to \infty} \left(\sum_{i=1}^{[nx]} \ln \left(\frac{R_{i+1} - A_{i+1} + A_i}{R_i} \right)^{\alpha(\beta_i)} \right. \right. \\ &\qquad \left. \left. + \sum_{i=1}^{[nx]} \ln[1 - c(\beta_i - \beta_{i+1})] \right) \right\}. \end{aligned} \tag{3.12}$$

Theorem 1 now follows from Lemmas 3.1 and 3.2. ∎

Finally, we remark that the proof is essentially unchanged for a cusp. Lemmas 2.1 through 2.6 deal only with the operator H_β or two adjacent cones and thus do not need any modification. As we examine the asymptotic expansions in Lemmas 2.7 and 2.8, note that R may tend to zero as $x \to 0$. However, since we are fixing $x > 0$ and letting $n \to \infty$, the estimates are still valid as we now interpret $G = O(F)$ as $G \le cF$ provided $n \ge n_o(x, \nu)$. The necessary modifications in Lemma 3.1 are straightforward: take $x > 0$ and divide the interval $[x, 1/2]$ into n segments and construct a sequence of cones as before.

4. PROOF OF THEOREM 3

We prove Theorem 3 by constructing a piecewise quadratic function. Lemmas 4.1–4.3 record information on the pieces and their integrals. Lemmas 4.4 and 4.5 provide the crucial example of a piecewise quadratic function such that $\int (f'^4/f)$ is infinite. However, the function will not be defined on the entire half line. The modification to the case at hand is done in the main proof. Note that because $|f'(t)|$ decreases to zero, it suffices to prove the result for $r = 4$ only.

Notation. For any two sequences of real numbers $\{v_n\}$ and $\{\xi_n\}$ strictly decreasing to zero, define

$$\begin{aligned} \delta(v)_n &= v_{n-1} - v_n \\ Y_n &= \frac{\xi_{n+1}^2 v_{n-1} - \xi_n^2 v_n}{\delta(v)_n} \\ w_n &= \frac{2\delta(v)_n}{\xi_n + \xi_{n+1}}. \end{aligned} \tag{4.1}$$

LEMMA 4.1. *Let $\{v_n\}_{n \ge 0}$ and $\{\xi_n\}_{n \ge 0}$ be strictly decreasing to zero. The polynomial*

$$p_n(t) = \left(\frac{\xi_n^2 - \xi_{n+1}^2}{4\delta(v)_n} \right) t^2 - \xi_n t + v_n + \delta(v)_n$$

satisfies

$$p_n'(0) = -\xi_n, \quad p_n'(w_n) = -\xi_{n+1}, \quad p_n(0) = v_{n-1}, \quad \textit{and} \quad p_n(w_n) = v_n.$$

With $t_n \equiv 1 + \sum_{j=1}^{n} w_j$, $n \geq 0$, *we have that*

$$f\colon \left(1, 1 + \sum_{1}^{\infty} w_n\right) \to (0, \infty)$$

$$f(t) = \sum_{n=0}^{\infty} p_n(t - t_n)\mathbf{1}_{\{t_n < t \leq t_{n+1}\}}(t)$$

is differentiable, strictly decreasing, and $|f'|$ *is strictly decreasing.*

Here, $\mathbf{1}_A$ is the indicator function, or characteristic function, of the set A.

Proof. The statements concerning the $\{p_n\}$ can be verified directly. f is continuous since $p_{n-1}(w_{n-1}) = v_{n-1} = p_n(0)$; it is differentiable at each t_n by design, hence differentiable everywhere. The other conclusions are valid since they are true on each piece. ∎

Remark. $\{p_n(t)\}$ is determined by $\{v_n\}$ and $\{\xi_n\}$ alone, and the properties of f depend solely on the condition that $\{v_n\}$ and $\{\xi_n\}$ be strictly decreasing. Note that the domain of f depends only on the choice of $\{v_n\}$ and $\{\xi_n\}$ as well.

Elementary calculations give the following two lemmas.

LEMMA 4.2. *Let* $a, b, c \in \mathbf{R}$ *such that* $q \equiv b^2 - 4ac > 0$. *Then*

$$\int \frac{(b + 2cx)^4}{a + bx + cx^2}\,dx + \text{Const.}$$
$$= \frac{2}{3}(b + 2cx)^3 + 2q(b + 2cx) + q^{3/2}\log\left|\frac{b + 2cx - \sqrt{q}}{b + 2cx + \sqrt{q}}\right|.$$

LEMMA 4.3. *If* $Y_n > 0$,

$$\int_0^{w_n} \frac{p_n'(t)^4}{p_n(t)}\,dt = \frac{2}{3}(\xi_n^3 - \xi_{n+1}^3) + 2Y_n(\xi_n - \xi_{n+1})$$
$$+ Y_n^{3/2}\log\left|\frac{\xi_n\xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n\xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}\right|.$$

LEMMA 4.4. *Set* $\xi_n = n^{-1/4}$, $v_0 = 1$, $v_1 < 1$, *and* $v_n = v_1\prod_{j=2}^{n}(\log j)/(\log j + 1)$, $n \geq 2$. *Then* $Y_n > 0$ *for* n *sufficiently large, and the corresponding series* $\sum w_n$ *is convergent.*

Proof. Let $p > 0$ and define

$$g(x) = \left(\frac{x}{x+1}\right)^p \frac{\log x + 1}{\log x}.$$

We wish to show $g(x) > 1$ for large enough x. First observe

$$\lim_{x \to \infty} g(x) = 1.$$

Now,

$$g'(x) = \frac{p}{(x+1)^2}\left(\frac{x}{x+1}\right)^{p-1}\left(1 + \frac{1}{\log x}\right) - \frac{1}{n(\log n)^2}\left(\frac{x}{x+1}\right)^p.$$

Some rearranging shows

$$g'(x) < 0 \Leftrightarrow (\log x)^2 + \log x < \frac{1}{p}(x+1),$$

which is certainly true for large enough x (depending on p). Observe that $g(n) > 1$ implies

$$\frac{\log n + 1}{(n+1)^p} - \frac{\log n}{n^p} > 0.$$

Note

$$\frac{\delta(v)_n}{v_n} = \frac{1}{\log n},$$

so using the fact $v_{n-1} = \delta(v)_n + v_n$, we have

$$Y_n = \frac{\log n + 1}{\sqrt{n+1}} - \frac{\log n}{\sqrt{n}}. \tag{4.2}$$

By the above argument with $p = 1/2$, we have that $Y_n > 0$ for n sufficiently large.

Applying the above argument with $p = 2$ shows that for n large enough,

$$\frac{\log n + 1}{(n+1)^2} - \frac{\log n}{n^2} > 0 \Leftrightarrow \frac{1 + 1/\log n}{(n+1)^2} - \frac{1}{n^2} > 0$$

$$\Leftrightarrow \frac{v_n + \delta(v)_n}{v_n} > \left(\frac{n+1}{n}\right)^2$$

$$\Leftrightarrow v_n < \left(\frac{n}{n+1}\right)^2 v_{n-1}.$$

By iterating we see

$$v_n < \frac{\bar{c}}{n^2}$$

for some constant $\bar{c}$. Hence $\sum v_n$ as defined in the statement of the lemma is a convergent series. Write "$\sim$" for "converges/diverges with." From the definition of w_n (see (4.1)), we have

$$\sum w_n \sim \sum n^{1/4}\delta(v)_n < \sum n\,\delta(v)_n \sim \sum v_n < \infty.$$

and so $\sum w_n$ is convergent. ∎

LEMMA 4.5. *Let* $\{v_n\}$ *and* $\{\xi_n\}$ *be defined as in Lemma* 4.4. *Then*

$$\int_1^{1+\sum_1^\infty w_n} \frac{f'(t)^4}{f(t)}\,dt = \infty.$$

Proof. One may verify that

$$\frac{2}{3}\left(\xi_n^3 - \xi_{n+1}^3\right) = \frac{1}{2n^{7/4}} + O(n^{-11/4})$$

$$2Y_n(\xi_n - \xi_{n+1}) = \frac{\log n}{n^{7/4}} + O(n^{-7/4}).$$

Hence, to show that the integral diverges, we shall find

$$\begin{aligned} Y_n^{3/2} \log&\left|\frac{\xi_n\xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n\xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}\right| \\ &= Y_n^{3/2}\log\left|\frac{1 - \left(n^{1/4} - (n+1)^{1/4}\right)\sqrt{Y_n} - (n(n+1))^{1/4}Y_n}{1 + \left(n^{1/4} - (n+1)^{1/4}\right)\sqrt{Y_n} - (n(n+1))^{1/4}Y_n}\right| \\ &= O\left(\frac{\log(\log n + 1) - \log\log n}{n^{3/4}}\right). \end{aligned} \tag{4.3}$$

Step 1.

$$\frac{1 - \left(n^{1/4} - (n+1)^{1/4}\right)\sqrt{Y_n} - (n(n+1))^{1/4}Y_n}{1 + \left(n^{1/4} - (n+1)^{1/4}\right)\sqrt{Y_n} - (n(n+1))^{1/4}Y_n} > 1.$$

This statement is equivalent to

$$\left(n^{1/4} - (n+1)^{1/4}\right) < -\left(n^{1/4} - (n+1)^{1/4}\right).$$

But $(n^{1/4} - (n+1)^{1/4}) < 0 < -(n^{1/4} - (n+1)^{1/4})$. Hence we can remove the absolute value in (4.3). Recall that Y_n is given by (4.2).

Step 2. $\lim_{n\to\infty} n^{1/2} Y_n = 1$.
This is straightforward,

$$\begin{aligned}\lim_{n\to\infty} n^{1/2} Y_n &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/2} - \lim_{n\to\infty}\frac{1-\left(n/(n+1)\right)^{1/2}}{1/\log n}\\ &= 1 + \frac{1}{2}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1/2}\frac{n(\log n)^2}{(n+1)^2}\\ &= 1,\end{aligned}$$

by an application of l'Hôpital's rule.

Step 3. $\lim_{n\to\infty}(2n/(\log n + 1))(1 - (n^{1/4} - (n+1)^{1/4})\sqrt{Y_n} - (n(n+1))^{1/4} Y_n) = 1$.
Let $\alpha_n = n^{1/4} - (n+1)^{1/4}$. We first simplify a bit:

$$\begin{aligned}&1 - \alpha_n\sqrt{Y_n} - (n(n+1))^{1/4} Y_n\\ &\quad = 1 - \alpha_n\sqrt{Y_n} - (n(n+1))^{1/4}\left[\frac{\log n + 1}{\sqrt{n+1}} - \frac{\log n}{\sqrt{n}}\right]\\ &\quad = \frac{n^{1/2}(n+1)^{1/2} - n^{3/4}(n+1)^{1/4}}{\sqrt{n(n+1)}}\\ &\qquad + \frac{n^{1/4}(n+1)^{3/4} - n^{3/4}(n+1)^{1/4}}{\sqrt{n(n+1)}}\log n - \alpha_n\sqrt{Y_n}\\ &\quad = \frac{(n+1)^{1/4} - n^{1/4}}{(n+1)^{1/4}} + \frac{\sqrt{n+1} - \sqrt{n}}{n^{1/4}(n+1)^{1/4}}\log n - \alpha_n\sqrt{Y_n}.\end{aligned}$$

Now,

$$\frac{(n+1)^{1/4}-n^{1/4}}{(n+1)^{1/4}} \sim \frac{1}{4n^{3/4}(n+1)^{1/4}} \sim \frac{1}{4n}$$

$$\frac{\sqrt{n+1}-\sqrt{n}}{n^{1/4}(n+1)^{1/4}} \sim \frac{1}{2\sqrt{n}\,(n(n+1))^{1/4}} \sim \frac{1}{2n},$$

since for $0 < p < 1$,

$$\lim_{n\to\infty} n^{1-p}\left((n+1)^{p}-n^{p}\right) = p. \tag{4.4}$$

Hence,

$$\frac{2n}{\log n+1}\,\frac{(n+1)^{1/4}-n^{1/4}}{(n+1)^{1/4}} \sim \frac{1}{2(\log n+1)} \to 0$$

$$\frac{2n}{\log n+1}\,\frac{\sqrt{n+1}-\sqrt{n}}{n^{1/4}(n+1)^{1/4}}\log n \to 1.$$

From Step 2,

$$\alpha_n\sqrt{Y_n}\,\frac{2n}{\log n+1} \sim \frac{1}{4n^{3/4}}\,\frac{1}{n^{1/4}}\,\frac{2n}{\log n+1} \to 0,$$

whence the statement of Step 3.

Step 4. $\lim_{n\to\infty}(2n/\log n)(1+(n^{1/4}-(n+1)^{1/4})\sqrt{Y_n}-(n(n+1))^{1/4}Y_n) = 1$.

As in the last step,

$$\begin{aligned} 1 &+ \alpha_n\sqrt{Y_n} - (n(n+1))^{1/4}Y_n \\ &= \frac{(n+1)^{1/4}-n^{1/4}}{(n+1)^{1/4}} + \frac{\sqrt{n+1}-\sqrt{n}}{n^{1/4}(n+1)^{1/4}}\log n + \alpha_n\sqrt{Y_n}\,. \end{aligned}$$

The same argument in Step 3 works here as well.

Step 5.

$$\lim_{n\to\infty}\frac{1-\left(n^{1/4}-(n+1)^{1/4}\right)\sqrt{Y_n}-(n(n+1))^{1/4}Y_n}{1+\left(n^{1/4}-(n+1)^{1/4}\right)\sqrt{Y_n}-(n(n+1))^{1/4}Y_n}\,\frac{\log n}{\log n+1} = 1.$$

This is evident from Steps 3 and 4.

Step 6.

$$\lim_{n\to\infty} Y_n^{3/2} \log\left(\frac{1-\left(n^{1/4}-(n+1)^{1/4}\right)\sqrt{Y_n}-\left(n(n+1)\right)^{1/4}Y_n}{1+\left(n^{1/4}-(n+1)^{1/4}\right)\sqrt{Y_n}-\left(n(n+1)\right)^{1/4}Y_n}\right)$$
$$\times \frac{n^{3/4}}{\log(\log n+1)-\log\log n}=1.$$

From Steps 2 and 5, we have that the limit in question is equal to

$$\lim_{n\to\infty} \frac{1}{n^{3/4}} \log\left(\frac{\log n+1}{\log n}\right) \frac{n^{3/4}}{\log(\log n+1)-\log\log n},$$

which is clearly 1.

After all this, we may conclude that

$$\sum_{n=1}^{\infty} \int_0^{w_n} \frac{p_n'(t)^4}{p_n(t)}\, dt \sim \sum_{n=3}^{\infty} \frac{\log(\log n+1)-\log\log n}{n^{3/4}}.$$

This sum on the right is apparently infinite:

$$\begin{aligned}
&\int_3^{\infty} \frac{\log(\log x+1)-\log\log x}{x^{3/4}}\, dx \\
&\qquad \geq \int_3^{\infty} \frac{\log(\log x+1)-\log\log x}{x}\, dx \\
&\qquad = \int_{\log 3}^{\infty} (\log(u+1)-\log u)\, du \\
&\qquad = \left(\log u+u(\log(u+1)-\log u)-1\right)\Big|_{\log 3}^{\infty} \\
&\qquad = \infty. \quad \blacksquare
\end{aligned}$$

To finally construct our example, we make use of the implication of Lemmas 4.4 and 4.5: we can get to infinity in a finite amount of time.

Proof of Theorem 3. Fix $\{\xi_n\}$ as in Lemmas 4.4 and 4.5. By Lemma 4.1, it suffices to choose $\{v_n\}$, strictly decreasing to zero such that

$$\begin{cases} \displaystyle\sum_{n=0}^{\infty} w_n = \infty \\ \displaystyle\sum_{n=1}^{\infty} \int_0^{w_n} \frac{p_n'(t)^4}{p_n(t)}\, dt = \infty. \end{cases}$$

From (4.1) and (4.4) we conclude $\sum w_n \sim \sum n^{1/4}\delta(v)_n \sim \sum n^{-3/4} v_n$. Thus, our strategy is to insert terms into the sequence $\{v_n\}$ of Lemmas 4.4 and 4.5 to make $\sum n^{-3/4} v_n = \infty$.

Let $v_0 = 1$, $v_1 = 15/16$, and $v_n = v_1 \prod_{j=2}^{n} (\log j)/\log j + 1)$, for $2 \le n \le N_1$, where N_1 is chosen so that

$$\sum_{n=1}^{N_1} Y_n^{3/2} \log \left| \frac{\xi_n \xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n \xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n} \right| > 2^1.$$

For $N_1 + 1 \le n \le N_2$, let $v_n = (j_1 + (n - N_1))^{-1/4}$, where $j_1 = (\lceil v_{N_1}^{-4} \rceil$ ($\lceil \cdot \rceil$ is the greatest integer function) and N_2 is chosen so that

$$\sum_{n=N_1+1}^{N_2} n^{-3/4} v_n > 2^1.$$

Observe $v_{N_1+1} < v_{N_1}$. For $N_2 + 1 \le n \le N_3$, let $v_n = (15/16) v_{N_2} \times \prod_{j=2}^{n} (\log j/\log j + 1)$, where N_3 is chosen so that

$$\sum_{n=N_2+1}^{N_3} Y_n^{3/2} \log \left| \frac{\xi_n \xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n \xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n} \right| > 2^2.$$

We continue in this manner, obtaining a sequence $\{N_l\}_{l \ge 1}$ such that

$$\begin{cases} j_{2k-1} = \left\lceil v_{N_{2k-1}}^{-4} \right\rceil \\ v_n = \left(j_{2k-1} + n - N_{2k-1} \right)^{-1/4}, \quad N_{2k-1} + 1 \le n \le N_{2k} \\ \displaystyle\sum_{N_{2k-1}+1}^{N_{2k}} n^{-3/4} v_n > 2^k \end{cases}$$

$$\begin{cases} v_n = \dfrac{15}{16} v_{N_{2k}} \displaystyle\prod_{j=2}^{N} \dfrac{\log j}{\log j + 1}, \qquad N_{2k} + 1 \le n \le N_{2k+1} \\ \displaystyle\sum_{N_{2k}+1}^{N_{2k+1}} Y_n^{3/2} \log \left| \dfrac{\xi_n \xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n \xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n} \right| > 2^k. \end{cases}$$

Clearly the resulting sequence $\{v_n\}$ is strictly decreasing to zero and the series Σw_n is divergent since

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-3/4} v_n &\ge \sum_{k=1} \sum_{N_{2k-1}+1}^{N_{2k}} n^{-3/4} v_n \\ &\ge \sum_{k=1}^{\infty} 2^k \\ &= \infty. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_1^{\infty} \frac{f'(t)^4}{f(t)} \, dt &= \sum_{n=1}^{\infty} \int_0^{w_n} \frac{p_n'(t)^4}{p_n(t)} \, dt \\ &\ge \sum_{k=1}^{\infty} \sum_{N_{2k}+1}^{N_{2k+1}} Y_n^{3/2} \log \left| \frac{\xi_n \xi_{n+1} + (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n}{\xi_n \xi_{n+1} - (\xi_n - \xi_{n+1})\sqrt{Y_n} - Y_n} \right| \\ &\ge \sum_{k=1}^{\infty} 2^k \\ &= \infty. \quad \blacksquare \end{aligned}$$

Proof of Corollary 1. By a change of variable,

$$\int_0^{1/2} \frac{|\nu'(t)|^r}{\nu(t)} \, dt = \int_2^{\infty} \frac{|\nu'(1/t)|^r}{\nu(1/t)} \frac{dt}{t^2}.$$

Set $f(t) = \nu(1/t)$ for $0 < t < 1$, and observe

$$f \text{ satisfies (A1) and (A2)} \Leftrightarrow \nu \text{ satisfies (B1) and (B2)}.$$

Now,

$$|f'(t)|^r = \frac{1}{t^{2r-2}} \frac{|\nu'(1/t)|^r}{t^2},$$

so,

$$\int_0^{1/2} \frac{|\nu'(t)|^r}{\nu(t)}\, dt = \int_2^{\infty} \frac{t^{2r-2}|f'(t)|^r}{f(t)}\, dt$$
$$\geq \int_2^{\infty} \frac{|f'(t)|^r}{f(t)}\, dt.$$

The theorem implies the existence of a differentiable function f satisfying (A1), (A2), and (1.2), so it follows that there exists a differentiable function ν satisfying (B1), (B2), and (1.5). ∎

Remark. It appears that by stopping the derivative more often, i.e., taking $\xi_n = n^{-1/m}$ for higher m, one could use this construction to obtain functions for which $\int(|f'|^r/f)\, dt = \infty$, $r > 4$.

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